

ON MULTIPLE ZETA VALUES AND FINITE MULTIPLE ZETA VALUES OF MAXIMAL HEIGHT

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ABSTRACT. An explicit formula for the height-one multiple zeta values was proved by Kaneko and the second author. We give an alternative proof of this result and its generalization. We also prove its counterpart for the finite multiple zeta values.

1. INTRODUCTION/MAIN THEOREM

For positive integers $k_1, \dots, k_d \in \mathbb{N}$ with $k_1 \geq 2$, the multiple zeta value (MZV) is defined by

$$\zeta(k_1, \dots, k_d) := \sum_{n_1 > \dots > n_d \geq 1} \frac{1}{n_1^{k_1} \dots n_d^{k_d}}.$$

The quantities $\text{wt}(\mathbf{k}) := k_1 + \dots + k_d$, $\text{dep}(\mathbf{k}) := d$, and $\text{ht}(\mathbf{k}) := \#\{i \mid k_i \geq 2, 1 \leq i \leq d\}$ are called the weight, the depth, and the height of the index set $\mathbf{k} = (k_1, \dots, k_d)$ (or of the multiple zeta value $\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_d)$), respectively. For two indices $\mathbf{k} = (k_1, \dots, k_i)$ and $\mathbf{r} = (r_1, \dots, r_i)$ of the same depth, $\zeta(\mathbf{k} + \mathbf{r})$ denotes $\zeta(k_1 + r_1, \dots, k_i + r_i)$. In [4], M. Kaneko and the second author proved the following explicit formula for the height-one multiple zeta values.

Theorem 1.1 ([4]). *For $k, r \in \mathbb{N}$, we have*

$$\zeta(k+1, \underbrace{1, \dots, 1}_{r-1}) = \sum_{i=1}^{\min(k,r)} (-1)^{i-1} \sum_{\substack{\text{wt}(\mathbf{k})=k, \text{wt}(\mathbf{r})=r \\ \text{dep}(\mathbf{k})=\text{dep}(\mathbf{r})=i}} \zeta(\mathbf{k} + \mathbf{r}).$$

We note that the right-hand side of this formula is symmetric in r and k , and thus the formula makes the duality $\zeta(k+1, \underbrace{1, \dots, 1}_{r-1}) = \zeta(r+1, \underbrace{1, \dots, 1}_{k-1})$ visible. One of the aims of

this paper is to prove a generalization of Theorem 1.1. For two indices \mathbf{k}, \mathbf{k}' , we say \mathbf{k}' refines \mathbf{k} (denoted $\mathbf{k}' \succeq \mathbf{k}$) if \mathbf{k} can be obtained from \mathbf{k}' by combining some of its adjacent parts. For example, $(1, 2, 3, 4) \succeq (1+2, 3+4) = (3, 7)$. Our main theorem is the following:

Theorem 1.2. *For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, we have*

$$\sum_{\substack{r_1 + \dots + r_d = r \\ r_i \geq 1 (1 \leq i \leq d)}} \zeta(k_1+1, \underbrace{1, \dots, 1}_{r_1-1}, \dots, k_d+1, \underbrace{1, \dots, 1}_{r_d-1}) = \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \text{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\text{wt}(\mathbf{r})=r \\ \text{dep}(\mathbf{r})=\text{dep}(\mathbf{k}')}} (-1)^{\text{dep}(\mathbf{k}')-d} \zeta(\mathbf{k}' + \mathbf{r}).$$

The case $d = 1$ of Theorem 1.2 gives Theorem 1.1. In this paper, we will give two different proofs of Theorem 1.2. The first proof is based on the duality theorem and Ohno's relations,

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and the second is on the derivation relations. In section 4, we will also prove a counterpart of this theorem for the finite multiple zeta values.

Example 1.3. When $k = 3, r = 4$ in Theorem 1.1, or $\mathbf{k} = (3), r = 4$ in Theorem 1.2, we have

$$\begin{aligned}\zeta(4, 1, 1, 1) &= \zeta(7) - (\zeta(5, 2) + 2\zeta(4, 3) + 2\zeta(3, 4) + \zeta(2, 5)) \\ &\quad + \zeta(3, 2, 2) + \zeta(2, 3, 2) + \zeta(2, 2, 3).\end{aligned}$$

The following is another example of Theorem 1.2.

Example 1.4. When $\mathbf{k} = (3, 2), r = 4$ in Theorem 1.2, we have

$$\begin{aligned}&\zeta(4, 1, 1, 3) + \zeta(4, 1, 3, 1) + \zeta(4, 3, 1, 1) \\ &= \zeta(6, 3) + \zeta(5, 4) + \zeta(4, 5) \\ &\quad - (\zeta(5, 2, 2) + \zeta(4, 3, 2) + 2\zeta(4, 2, 3) + 2\zeta(3, 3, 3) + \zeta(3, 2, 4) + \zeta(2, 4, 3) + \zeta(2, 3, 4)) \\ &\quad + \zeta(3, 2, 2, 2) + \zeta(2, 3, 2, 2) + \zeta(2, 2, 2, 3).\end{aligned}$$

Remark 1.5. S. Yamamoto also obtained a generalization of Theorem 1.1 by using generating function [11]. On multiple sums of multiple zeta values, unpublished manuscript (in Japanese). (2016)

2. PROOF OF MAIN THEOREM

2.1. The duality theorem and Ohno's relations for the MZVs. We recall some known results that will be used in the proof of Theorem 1.2.

Definition 1. Let $\mathbf{k} = (k_1, \dots, k_d)$ be an index set with $k_1 \geq 2$. We write

$$\mathbf{k} = (a_1 + 1, \underbrace{1, \dots, 1}_{b_1 - 1}, \dots, a_s + 1, \underbrace{1, \dots, 1}_{b_s - 1})$$

with $a_p, b_q \geq 1$. Then, we define the dual index set of \mathbf{k} as

$$\mathbf{k}^* = (b_s + 1, \underbrace{1, \dots, 1}_{a_s - 1}, \dots, b_1 + 1, \underbrace{1, \dots, 1}_{a_1 - 1}).$$

The following result, which is a direct consequence of the iterated integral expression, provides the so-called duality theorem for the MZVs.

Theorem 2.1. For $\mathbf{k} = (k_1, \dots, k_d)$ with $k_1 \geq 2$, we have

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^*).$$

Ohno's relations is one of the most general explicit relations among the MZVs. This is a generalization of both the duality theorem above and the well-known sum formula.

Theorem 2.2 (Ohno [8]). For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ with $k_1 \geq 2$ and $m \in \mathbb{Z}_{\geq 0}$, we have

$$\sum_{\substack{\varepsilon_1 + \dots + \varepsilon_d = m \\ \varepsilon_i \geq 0 (1 \leq i \leq d)}} \zeta(k_1 + \varepsilon_1, k_2 + \varepsilon_2, \dots, k_d + \varepsilon_d) = \sum_{\substack{\varepsilon'_1 + \dots + \varepsilon'_{d'} = m \\ \varepsilon'_i \geq 0 (1 \leq i \leq d')}} \zeta(k'_1 + \varepsilon'_1, k'_2 + \varepsilon'_2, \dots, k'_{d'} + \varepsilon'_{d'}),$$

where $(k'_1, \dots, k'_{d'})$ is the dual of \mathbf{k} .

2.2. **Proof of Theorem 1.2.** We set $\text{wt}(\mathbf{k}) := k$.

$$\begin{aligned}
R.H.S. &= \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \text{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\text{wt}(\mathbf{r})=r \\ \text{dep}(\mathbf{r})=\text{dep}(\mathbf{k}')}} (-1)^{\text{dep}(\mathbf{k}')-d} \zeta(\mathbf{k}' + \mathbf{r}) \\
&= \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{\substack{(k'_1, \dots, k'_j) \succeq \mathbf{k} \\ r_1 + \dots + r_j = r \\ r_i \geq 1 \ (1 \leq i \leq j)}} \zeta(k'_1 + r_1, \dots, k'_j + r_j).
\end{aligned}$$

From Ohno's relations, we have

$$\begin{aligned}
&\sum_{\substack{r_1 + \dots + r_j = r \\ r_i \geq 1 \ (1 \leq i \leq j)}} \zeta(k'_1 + r_1, \dots, k'_j + r_j) \\
&= \sum_{\substack{r_1 + \dots + r_j = r-j \\ r_i \geq 0 \ (1 \leq i \leq j)}} \zeta(k'_1 + r_1 + 1, \dots, k'_j + r_j + 1) \\
&= \sum_{\substack{r_1 + \dots + r_k = r-j \\ r_i \geq 0 \ (1 \leq i \leq k)}} \zeta(r_1 + 2, \underbrace{r_2 + 1, \dots, r_{k'_j} + 1}_{k'_j - 1}, \dots, r_{k-k'_1+1} + 2, \underbrace{r_{k-k'_1+2} + 1, \dots, r_k + 1}_{k'_1 - 1}).
\end{aligned}$$

Then,

$$\begin{aligned}
(1) \quad R.H.S. &= \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{\substack{(k'_1, \dots, k'_j) \succeq \mathbf{k} \\ r_1 + \dots + r_k = r-j \\ r_i \geq 0 \ (1 \leq i \leq k)}} \sum_{\substack{r_1 + \dots + r_{k'_j} = r-j \\ r_i \geq 0 \ (1 \leq i \leq k'_j)}} \zeta(r_1 + 2, \underbrace{r_2 + 1, \dots, r_{k'_j} + 1}_{k'_j - 1}, \dots, \\
&\quad \dots, r_{k-k'_1+1} + 2, \underbrace{r_{k-k'_1+2} + 1, \dots, r_k + 1}_{k'_1 - 1}).
\end{aligned}$$

Now, for a fixed $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, and a variable $s \in \mathbb{N}$, we set

$$\begin{aligned}
F(s) &:= F_{\mathbf{k},r}(s) \\
&:= \sum_{\substack{r_1 + \dots + r_k = r-d \\ r_i \geq 0 \ (1 \leq i \leq k), \text{ht}=s}} \zeta(r_1 + 2, \underbrace{r_2 + 1, \dots, r_{k_d} + 1}_{k_d - 1}, \dots, r_{k-k_1+1} + 2, \underbrace{r_{k-k_1+2} + 1, \dots, r_k + 1}_{k_1 - 1}),
\end{aligned}$$

where the sum runs over those r_i s such that the argument of ζ is of height s . For each j in equation (1), the number of each height s MZV is $\binom{s-d}{j-d}$ because it is determined by the partitions of \mathbf{k} and the allocation of r . (Here, we note that there are $s-d$ places with a component greater than 1 that are determined by the partitions of \mathbf{k} and the allocation of r , and there are $j-d$ places which depend on the various partitions of \mathbf{k} .) Thus, by focusing on the height and retake the sums, we have

$$R.H.S. = \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{s=j}^{\min(k,r)} \binom{s-d}{j-d} F(s).$$

From the binomial theorem and the duality theorem, we have

$$\begin{aligned}
R.H.S. &= \binom{0}{0} F(d) + \left(\binom{1}{0} - \binom{1}{1} \right) F(d+1) + \cdots + \left(\sum_{m=0}^{\min(k,r)-d} (-1)^m \binom{\min(k,r)-d}{m} \right) F(\min(k,r)) \\
&= F(d) \\
&= \sum_{\substack{r_1+\cdots+r_d=r-d \\ r_i \geq 0 (1 \leq i \leq d)}} \zeta(r_1+2, \underbrace{1, \dots, 1}_{k_d-1}, \dots, r_d+2, \underbrace{1, \dots, 1}_{k_1-1}) \\
&= \sum_{\substack{r_1+\cdots+r_d=r \\ r_i \geq 1 (1 \leq i \leq d)}} \zeta(r_1+1, \underbrace{1, \dots, 1}_{k_d-1}, \dots, r_d+1, \underbrace{1, \dots, 1}_{k_1-1}) \\
&= \sum_{\substack{r_1+\cdots+r_d=r \\ r_i \geq 1 (1 \leq i \leq d)}} \zeta(k_1+1, \underbrace{1, \dots, 1}_{r_1-1}, \dots, k_d+1, \underbrace{1, \dots, 1}_{r_d-1}) \\
&= L.H.S.
\end{aligned}$$

3. ALTERNATIVE PROOF

3.1. Algebraic setup and the derivation relations. In this section, we give an alternative proof of Theorem 1.2. Throughout the proof, we use the algebraic setup introduced by M. Hoffman in [1]. Let $\mathfrak{H} := \mathbb{Q}\langle x, y \rangle$ be the noncommutative polynomial ring in two indeterminates x, y , and \mathfrak{H}^1 (resp. \mathfrak{H}^0) its subrings $\mathbb{Q} + \mathfrak{H}y$ (resp. $\mathbb{Q} + x\mathfrak{H}y$). We set $z_k := x^{k-1}y$ ($k \in \mathbb{N}$). Then \mathfrak{H}^1 is freely generated by $\{z_k\}_{k \geq 1}$. We define the \mathbb{Q} -linear map $Z : \mathfrak{H}^0 \rightarrow \mathbb{R}$ by $Z(1) := 1$ and $Z(z_{k_1} \cdots z_{k_d}) := \zeta(k_1, \dots, k_d)$.

A derivation ∂ on \mathfrak{H} is a \mathbb{Q} -linear endomorphism of \mathfrak{H} satisfying Leibniz's rule $\partial(wv) = \partial(w)v + w\partial(v)$. Such a derivation is uniquely determined by its images of generators x and y . Set $z := x + y$. For each $l \in \mathbb{N}$, the derivation ∂_l on \mathfrak{H} is defined by $\partial_l(x) := xz^{l-1}y$ and $\partial_l(y) := -xz^{l-1}y$. We note that $\partial_l(1) = 0$ and $\partial_l(z) = 0$. K. Ihara, Kaneko and D. Zagier proved the derivation relations for the MZVs.

Theorem 3.1 (Ihara–Kaneko–Zagier [3]). *For $l \in \mathbb{N}$, we have*

$$Z(\partial_l(w)) = 0 \quad (w \in \mathfrak{H}^0).$$

Now, we prepare some additional notation. Let α be the endomorphism on \mathfrak{H} such that $\alpha(x) := x - xy$ and $\alpha(y) := -xy$, and τ be the anti-automorphism on \mathfrak{H} such that $\tau(x) := y$ and $\tau(y) := x$. For each $l \in \mathbb{N}$, we define the derivation D_l on \mathfrak{H} by $D_l(x) := 0$ and $D_l(y) := x^l y$. Set $\sigma := \exp\left(\sum_{l=1}^{\infty} \frac{D_l}{l}\right)$. Then, we find the map σ is an automorphism on $\widehat{\mathfrak{H}} = \mathbb{Q}\langle\langle x, y \rangle\rangle$, and we see that $\sigma(x) = x$ and $\sigma(y) = \frac{1}{1-x}y$. (See also [3, Section 6] and [10, Appendix].)

3.2. Alternative proof of Theorem 1.2. According to Ihara, Kaneko and Zagier in [3, Proof of Theorem 3], we first note that

$$(2) \quad \sigma - \tau\sigma\tau = \left(1 - \exp\left(\sum_{l=1}^{\infty} \frac{\partial_l}{l}\right)\right) \sigma.$$

By Theorem 3.1, we have

$$Z\left(\left(1 - \exp\left(\sum_{l=1}^{\infty} \frac{\partial_l}{l}\right)\right) \sigma(w)\right) = 0 \quad (w \in \mathfrak{H}^0).$$

From the equality (2) and by putting $\alpha(x^{k_1-1}y \cdots x^{k_d-1}y)$ into w ,

$$Z((\sigma - \tau\sigma\tau)\alpha(x^{k_1-1}y \cdots x^{k_d-1}y)) = 0.$$

Here,

$$\begin{aligned} \sigma\alpha(x^{k_1-1}y \cdots x^{k_d-1}y) &= \sigma((x - xy)^{k_1-1}(-xy) \cdots (x - xy)^{k_d-1}(-xy)) \\ &= (-1)^d \sigma((x - xy)^{k_1-1}xy \cdots (x - xy)^{k_d-1}xy) \\ &= (-1)^d \left(x - \frac{x}{1-x}y\right)^{k_1-1} \frac{x}{1-x}y \cdots \left(x - \frac{x}{1-x}y\right)^{k_d-1} \frac{x}{1-x}y \\ &= (-1)^d \sum_{r=d}^{\infty} \sum_{\substack{e_{1,1}+\cdots+e_{d,k_d}=r \\ e_{i,j} \geq 0 \ (1 \leq j \leq k_i-1) \\ e_{i,j} \geq 1 \ (j=k_i)}} (-x^{e_{1,1}}y) \cdots (-x^{e_{1,k_1-1}}y)x^{e_{1,k_1}}y \\ &\quad \cdots \cdots \cdots (-x^{e_{d,1}}y) \cdots (-x^{e_{d,k_d-1}}y)x^{e_{d,k_d}}y. \end{aligned}$$

When $e_{i,j} = 0$, we understand $x^{e_{i,j}}y = -x$. On the other hand,

$$\begin{aligned} \tau\sigma\tau\alpha(x^{k_1-1}y \cdots x^{k_d-1}y) &= \tau\sigma\tau((x - xy)^{k_1-1}(-xy) \cdots (x - xy)^{k_d-1}(-xy)) \\ &= (-1)^d \tau\sigma\tau((x - xy)^{k_1-1}xy \cdots (x - xy)^{k_d-1}xy) \\ &= (-1)^d \tau\sigma(xy(y - xy)^{k_d-1} \cdots xy(y - xy)^{k_1-1}) \\ &= (-1)^d \tau \left(\frac{x}{1-x}y^{k_d} \cdots \frac{x}{1-x}y^{k_1} \right) \\ &= (-1)^d x^{k_1} \frac{y}{1-y} \cdots x^{k_d} \frac{y}{1-y} \\ &= (-1)^d \sum_{r=d}^{\infty} \sum_{\substack{r_1+\cdots+r_d=r \\ r_i \geq 1 \ (1 \leq i \leq d)}} x^{k_1}y^{r_1} \cdots x^{k_d}y^{r_d}. \end{aligned}$$

Then, we have

$$\begin{aligned} &\sum_{r=d}^{\infty} \sum_{\substack{e_{1,1}+\cdots+e_{d,k_d}=r \\ e_{i,j} \geq 0 \ (1 \leq j \leq k_i-1) \\ e_{i,j} \geq 1 \ (j=k_i)}} (-x^{e_{1,1}}y) \cdots (-x^{e_{1,k_1-1}}y)x^{e_{1,k_1}}y \\ &\quad \cdots \cdots \cdots (-x^{e_{d,1}}y) \cdots (-x^{e_{d,k_d-1}}y)x^{e_{d,k_d}}y \\ &= \sum_{r=d}^{\infty} \sum_{\substack{r_1+\cdots+r_d=r \\ r_i \geq 1 \ (1 \leq i \leq d)}} x^{k_1}y^{r_1} \cdots x^{k_d}y^{r_d}. \end{aligned}$$

This finishes the proof of the theorem.

4. THE COUNTERPART OF THEOREM 1.2 FOR THE FINITE MULTIPLE ZETA VALUES

4.1. Definitions and second main result. In this section, we prove the counterpart of Theorem 1.2 for what we call ‘finite multiple zeta values (FMZVs)’, a generic term for the \mathcal{A} -finite multiple zeta values and the symmetrized multiple zeta values.

We consider the collection of truncated sums $\zeta_p(k_1, \dots, k_d) := \sum_{p > n_1 > \cdots > n_d \geq 1} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} \pmod{\text{ulo all primes } p}$ in the quotient ring $\mathcal{A} = (\prod_p \mathbb{Z}/p\mathbb{Z})/(\bigoplus_p \mathbb{Z}/p\mathbb{Z})$, which is a \mathbb{Q} -algebra. Elements of \mathcal{A} are represented by $(a_p)_p$, where $a_p \in \mathbb{Z}/p\mathbb{Z}$, and two elements $(a_p)_p$ and $(b_p)_p$ are

identified if and only if $a_p = b_p$ for all but finitely many primes p . For integers $k_1, \dots, k_d \in \mathbb{N}$, the \mathcal{A} -finite multiple zeta value $\zeta_{\mathcal{A}}(k_1, \dots, k_d)$ is defined by

$$\zeta_{\mathcal{A}}(k_1, \dots, k_d) := \left(\sum_{p > n_1 > \dots > n_d \geq 1} \frac{1}{n_1^{k_1} \dots n_d^{k_d}} \bmod p \right)_p \in \mathcal{A}.$$

The symmetrized multiple zeta values was first introduced by Kaneko and Zagier in [5, 6]. For integers $k_1, \dots, k_d \in \mathbb{N}$, we let

$$\zeta_{\mathcal{S}}^*(k_1, \dots, k_d) := \sum_{i=0}^d (-1)^{k_1 + \dots + k_i} \zeta^*(k_i, \dots, k_1) \zeta^*(k_{i+1}, \dots, k_d).$$

Here, the symbols ζ^* on the right-hand sides stand for the regularized values coming from harmonic regularizations, i.e., real values obtained by taking constant terms of harmonic regularizations as explained in [3]. In the sum, we understand $\zeta^*(\emptyset) = 1$. Let $\mathcal{Z}_{\mathbb{R}}$ be the \mathbb{Q} -vector subspace of \mathbb{R} spanned by 1 and all MZVs, which is a \mathbb{Q} -algebra. Then, the symmetrized multiple zeta value $\zeta_{\mathcal{S}}(k_1, \dots, k_d)$ is defined as an element in the quotient ring $\mathcal{Z}_{\mathbb{R}}/\zeta(2)$ by

$$\zeta_{\mathcal{S}}(k_1, \dots, k_d) := \zeta_{\mathcal{S}}^*(k_1, \dots, k_d) \bmod \zeta(2).$$

(For more details, see [5, 6].)

Theorem 4.1. *For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, we have*

$$\begin{aligned} & \sum_{\substack{r_0 + \dots + r_d = r+1 \\ r_i \geq 1 \ (0 \leq i \leq d)}} \zeta_{\mathcal{F}}(\underbrace{1, \dots, 1}_{r_0-1}, k_1+1, \underbrace{1, \dots, 1}_{r_1-1}, \dots, \underbrace{1, \dots, 1}_{r_{d-1}-1}, k_d+1, \underbrace{1, \dots, 1}_{r_d-1}) \\ &= \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \text{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\text{wt}(\mathbf{r})=r \\ \text{dep}(\mathbf{r})=\text{dep}(\mathbf{k}')}} (-1)^{\text{dep}(\mathbf{k}')-d} \zeta_{\mathcal{F}}(\mathbf{k}' + \mathbf{r}) \quad (\mathcal{F} = \mathcal{A} \text{ or } \mathcal{S}). \end{aligned}$$

Remark 4.2. Denoting $\mathcal{Z}_{\mathcal{A}}$ by the \mathbb{Q} -vector subspace of \mathcal{A} spanned by 1 and all \mathcal{A} -finite multiple zeta values, Kaneko and Zagier conjecture that $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}_{\mathbb{R}}/\zeta(2)$ are isomorphic as \mathbb{Q} -algebras.

4.2. Proof of Theorem 4.1.

4.2.1. *Ohno type relations for the FMZVs.* K. Oyama proved Ohno type relations for the FMZVs, which was first conjectured by Kaneko in [5].

Definition 2. For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, we define Hoffman's dual of \mathbf{k} by

$$\mathbf{k}^{\vee} := (\underbrace{1, \dots, 1}_{k_1} + \underbrace{1, \dots, 1}_{k_2} + 1, \dots, 1 + \underbrace{1, \dots, 1}_{k_d}).$$

Then, Ohno type relations is the following:

Theorem 4.3 (Oyama [9]). *For $(k_1, \dots, k_d) \in \mathbb{N}^d$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{\varepsilon_1 + \dots + \varepsilon_d = m \\ \varepsilon_i \geq 0 \ (1 \leq i \leq d)}} \zeta_{\mathcal{F}}(k_1 + \varepsilon_1, \dots, k_d + \varepsilon_d) = \sum_{\substack{\varepsilon'_1 + \dots + \varepsilon'_{d'} = m \\ \varepsilon'_i \geq 0 \ (1 \leq i \leq d')}} \zeta_{\mathcal{F}}((k'_1 + \varepsilon'_1, \dots, k'_{d'} + \varepsilon'_{d'})^{\vee}),$$

where $(k'_1, \dots, k'_{d'}) = (k_1, \dots, k_d)^{\vee}$ is Hoffman's dual of (k_1, \dots, k_d) .

4.2.2. *Proof of Theorem 4.1.* We can prove Theorem 4.1 in the same manner as in the proof of Theorem 1.2. From Ohno type relations, we have

$$\begin{aligned}
R.H.S. &= \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \text{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\text{wt}(\mathbf{r})=r \\ \text{dep}(\mathbf{r})=\text{dep}(\mathbf{k}')}} (-1)^{\text{dep}(\mathbf{k}')-d} \zeta_{\mathcal{F}}(\mathbf{k}' + \mathbf{r}) \\
&= \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{\substack{(k'_1, \dots, k'_j) \succeq \mathbf{k} \\ r_1 + \dots + r_j = r-j \\ r_i \geq 0 (1 \leq i \leq j)}} \sum \zeta_{\mathcal{F}}(k'_1 + r_1 + 1, \dots, k'_j + r_j + 1) \\
&= \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{\substack{(k'_1, \dots, k'_j) \succeq \mathbf{k} \\ r_0 + \dots + r_k = r-j \\ r_i \geq 0 (0 \leq i \leq k)}} \sum \zeta_{\mathcal{F}}(\underbrace{(r_0 + 1, \dots, r_{k'_1-1} + 1, r_{k'_1} + 2,}_{k'_1} \\
&\quad \underbrace{r_{k'_1+1} + 1, \dots, r_{k'_1+k'_2-1} + 1}_{k'_2-1}, \dots, \underbrace{r_{k-k'_j-1-k'_j+1} + 1, \dots, r_{k-k'_j-1} + 1}_{k'_{j-1}-1}, \\
&\quad r_{k-k'_j} + 2, \underbrace{r_{k-k'_j+1} + 1, \dots, r_k + 1}_{k'_j})^\vee).
\end{aligned}$$

For a fixed $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, and a variable $s \in \mathbb{N}$, we set

$$\begin{aligned}
G(s) := G_{\mathbf{k},r}(s) &:= \sum_{\substack{r_0 + \dots + r_k = r-d-2 \\ r_i \geq 0 (0 \leq i \leq k), \text{ht}=s}} \zeta_{\mathcal{F}}(\underbrace{(r_0 + 2, r_1 + 1, \dots, r_{k_1-1} + 1, r_{k_1} + 2,}_{k_1-1} \\
&\quad \underbrace{r_{k_1+1} + 1, \dots, r_{k_1+k_2-1} + 1}_{k_2-1}, \dots, \underbrace{r_{k-k_{d-1}-k_d+1} + 1, \dots, r_{k-k_{d-1}} + 1}_{k_{d-1}-1}, \\
&\quad r_{k-k_d} + 2, \underbrace{r_{k-k_d+1} + 1, \dots, r_{k-1} + 1}_{k_d-1}, r_k + 2)^\vee),
\end{aligned}$$

$$\begin{aligned}
G'(s) := G'_{\mathbf{k},r}(s) &:= \sum_{\substack{r_1 + \dots + r_k = r-d-1 \\ r_i \geq 0 (1 \leq i \leq k), \text{ht}=s}} \zeta_{\mathcal{F}}(\underbrace{(1, r_1 + 1, \dots, r_{k_1-1} + 1, r_{k_1} + 2,}_{k_1-1} \\
&\quad \underbrace{r_{k_1+1} + 1, \dots, r_{k_1+k_2-1} + 1}_{k_2-1}, \dots, \underbrace{r_{k-k_{d-1}-k_d+1} + 1, \dots, r_{k-k_{d-1}} + 1}_{k_{d-1}-1}, \\
&\quad r_{k-k_d} + 2, \underbrace{r_{k-k_d+1} + 1, \dots, r_{k-1} + 1}_{k_d-1}, r_k + 2)^\vee) \\
&+ \sum_{\substack{r_0 + \dots + r_{k-1} = r-d-1 \\ r_i \geq 0 (0 \leq i \leq k-1), \text{ht}=s}} \zeta_{\mathcal{F}}(\underbrace{(r_0 + 2, r_1 + 1, \dots, r_{k_1-1} + 1, r_{k_1} + 2,}_{k_1-1} \\
&\quad \underbrace{r_{k_1+1} + 1, \dots, r_{k_1+k_2-1} + 1}_{k_2-1}, \dots, \underbrace{r_{k-k_{d-1}-k_d+1} + 1, \dots, r_{k-k_{d-1}} + 1}_{k_{d-1}-1}, \\
&\quad r_{k-k_d} + 2, \underbrace{r_{k-k_d+1} + 1, \dots, r_{k-1} + 1}_{k_d-1}, 1)^\vee),
\end{aligned}$$

$$G''(s) := G''_{\mathbf{k},r}(s) := \sum_{\substack{r_1+\dots+r_{k-1}=r-d \\ r_i \geq 0 (1 \leq i \leq k-1), \text{ht}=s}} \zeta_{\mathcal{F}}((1, \underbrace{r_1+1, \dots, r_{k_1-1}+1}_{k_1-1}, r_{k_1}+2, \underbrace{r_{k_1+1}+1, \dots, r_{k_1+k_2-1}+1}_{k_2-1}, \dots, \underbrace{r_{k-k_{d-1}-k_d+1}+1, \dots, r_{k-k_d-1}+1}_{k_{d-1}-1}, r_{k-k_d}+2, \underbrace{r_{k-k_d+1}+1, \dots, r_{k-1}+1}_{k_d-1}, 1)^{\vee}),$$

where the sum runs over those r_i s such that the argument of $\zeta_{\mathcal{F}}$ is of height s . By concentrating on the height, and adding up all the terms, then re-arranging the sums; we have

$$\begin{aligned} R.H.S. &= \sum_{j=d+1}^{\min(k+1, r-1)} (-1)^{j-d-1} \sum_{s=j}^{\min(k+1, r-1)} \binom{s-d-1}{j-d-1} G(s) \\ &+ \sum_{j=d}^{\min(k, r-1)} (-1)^{j-d} \sum_{s=j}^{\min(k, r-1)} \binom{s-d}{j-d} G'(s) \\ &+ \sum_{j=d-1}^{\min(k-1, r-1)} (-1)^{j-d+1} \sum_{s=j}^{\min(k-1, r-1)} \binom{s-d+1}{j-d+1} G''(s) \\ &= G(d+1) + G'(d) + G''(d-1) \\ &= \sum_{\substack{r_0+\dots+r_d=r-d \\ r_i \geq 0 (0 \leq i \leq d)}} \zeta_{\mathcal{F}}((r_0+1, \underbrace{1, \dots, 1}_{k_1-1}, r_1+2, \dots, r_{d-1}+2, \underbrace{1, \dots, 1}_{k_d-1}, r_d+1)^{\vee}) \\ &= \sum_{\substack{r_0+\dots+r_d=r+1 \\ r_i \geq 1 (0 \leq i \leq d)}} \zeta_{\mathcal{F}}(\underbrace{1, \dots, 1}_{r_0-1}, k_1+1, \underbrace{1, \dots, 1}_{r_1-1}, \dots, \underbrace{1, \dots, 1}_{r_{d-1}-1}, k_d+1, \underbrace{1, \dots, 1}_{r_d-1}) \\ &= L.H.S. \end{aligned}$$

4.3. Alternative proof of Theorem 4.1.

4.3.1. *The derivation relations for the FMZVs.* The derivation relations for the FMZVs is conjectured by Oyama and proved by the first author in [7].

We define two \mathbb{Q} -linear maps $Z_{\mathcal{A}}: \mathfrak{H}^1 \rightarrow \mathcal{A}$ and $Z_{\mathcal{S}}: \mathfrak{H}^1 \rightarrow \mathcal{Z}_{\mathbb{R}}/\zeta(2)$ respectively by $Z_{\mathcal{A}}(1) := 1$ and $Z_{\mathcal{A}}(z_{k_1} \cdots z_{k_d}) := \zeta_{\mathcal{A}}(k_1, \dots, k_d)$, and $Z_{\mathcal{S}}(1) := 1$ and $Z_{\mathcal{S}}(z_{k_1} \cdots z_{k_d}) := \zeta_{\mathcal{S}}(k_1, \dots, k_d)$. We also define the \mathbb{Q} -linear operator L_x on \mathfrak{H} by $L_x := xw$ ($w \in \mathfrak{H}$).

Theorem 4.4. *For $l \in \mathbb{N}$, we have*

$$Z_{\mathcal{F}}(L_x^{-1} \partial_l L_x(w)) = 0 \quad (w \in \mathfrak{H}^1, \mathcal{F} = \mathcal{A} \text{ or } \mathcal{S}).$$

4.3.2. *Alternative proof of Theorem 4.1.* By Theorem 4.4, we have

$$Z_{\mathcal{F}} \left(L_x^{-1} \left(1 - \exp \left(\sum_{l=1}^{\infty} \frac{\partial_l}{l} \right) \right) L_x \sigma(w) \right) = 0 \quad (w \in \mathfrak{H}^1).$$

Since $L_x \sigma = \sigma L_x$,

$$Z_{\mathcal{F}} \left(L_x^{-1} \left(1 - \exp \left(\sum_{l=1}^{\infty} \frac{\partial_l}{l} \right) \right) \sigma L_x(w) \right) = 0 \quad (w \in \mathfrak{H}^1).$$

From the equality (2) and by putting $\alpha(x^{k_1-1}y \cdots x^{k_d-1}y)$ into w , we have

$$Z_{\mathcal{F}}(L_x^{-1}(\sigma - \tau\sigma\tau)L_x\alpha(x^{k_1-1}y \cdots x^{k_d-1}y)) = 0.$$

Since $L_x^{-1}\sigma L_x = \sigma$ and by the same calculation in subsection 3.2, we have

$$\begin{aligned} L_x^{-1}\sigma L_x\alpha(x^{k_1-1}y \cdots x^{k_d-1}y) &= \sigma\alpha(x^{k_1-1}y \cdots x^{k_d-1}y) \\ &= (-1)^d \sum_{r=d}^{\infty} \sum_{\substack{e_{1,1}+\cdots+e_{d,k_d}=r \\ e_{i,j} \geq 0 \ (1 \leq j \leq k_i-1) \\ e_{i,j} \geq 1 \ (j=k_i)}} (-x^{e_{1,1}}y) \cdots (-x^{e_{1,k_1-1}}y)x^{e_{1,k_1}}y \\ &\quad \cdots \cdots \cdots (-x^{e_{d,1}}y) \cdots (-x^{e_{d,k_d-1}}y)x^{e_{d,k_d}}y. \end{aligned}$$

When $e_{i,j} = 0$, we understand $x^{e_{i,j}}y = -x$. On the other hand,

$$\begin{aligned} L_x^{-1}\tau\sigma\tau L_x\alpha(x^{k_1-1}y \cdots x^{k_d-1}y) &= L_x^{-1}\tau\sigma\tau L_x((x-xy)^{k_1-1}(-xy) \cdots (x-xy)^{k_d-1}(-xy)) \\ &= (-1)^d L_x^{-1}\tau\sigma\tau L_x((x-xy)^{k_1-1}xy \cdots (x-xy)^{k_d-1}xy) \\ &= (-1)^d L_x^{-1}\tau\sigma(xy(y-xy)^{k_d-1} \cdots xy(y-xy)^{k_1-1}y) \\ &= (-1)^d L_x^{-1}\tau \left(\frac{x}{1-x}y^{k_d} \cdots \frac{x}{1-x}y^{k_1} \frac{1}{1-x}y \right) \\ &= (-1)^d \frac{1}{1-y}x^{k_1} \frac{y}{1-y} \cdots x^{k_d} \frac{y}{1-y} \\ &= (-1)^d \sum_{r=d}^{\infty} \sum_{\substack{r_0+\cdots+r_d=r+1 \\ r_i \geq 1 \ (0 \leq i \leq d)}} y^{r_0-1}x^{k_1}y^{r_1} \cdots x^{k_d}y^{r_d}. \end{aligned}$$

Then, we have

$$\begin{aligned} &\sum_{r=d}^{\infty} \sum_{\substack{e_{1,1}+\cdots+e_{d,k_d}=r \\ e_{i,j} \geq 0 \ (1 \leq j \leq k_i-1) \\ e_{i,j} \geq 1 \ (j=k_i)}} (-x^{e_{1,1}}y) \cdots (-x^{e_{1,k_1-1}}y)x^{e_{1,k_1}}y \\ &\quad \cdots \cdots \cdots (-x^{e_{d,1}}y) \cdots (-x^{e_{d,k_d-1}}y)x^{e_{d,k_d}}y \\ &= \sum_{r=d}^{\infty} \sum_{\substack{r_0+\cdots+r_d=r+1 \\ r_i \geq 1 \ (0 \leq i \leq d)}} y^{r_0-1}x^{k_1}y^{r_1} \cdots x^{k_d}y^{r_d}. \end{aligned}$$

This completes the proof of the theorem.

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